Problem 2: The Stirling Cycle and Regeneration

Some heat engines use *regeneration* to improve their efficiency. Regeneration means the return to the working body of a part of heat that is transferred to the cooler. We will call this amount of returned heat "the amount of heat to be recovered" and we will denote it as Q_R. Different schemes of regeneration are possible: 1) transmission of Q_R through a special heat exchanger (regenerator); 2) transmission of Q_R to the working body of the second heat machine connected "in parallel" to the first machine; 3) transmission of Q_R through a *heat pump* (such devices are also called refrigeration units - they work as heat machines with a "reverse" cycle).

Part I: Parallel Heat Machines with Regeneration

Let's consider the second scheme. For a *Stirling engine* the problem of low efficiency is very significant, so the use of regenerators is critical for such engines. The cyclic process in the working body (WB) of a Stirling engine can be described with good accuracy as a cycle consisting of two isochores and two isotherms.

Let's have two Stirling engines connected in parallel. They get heat from a common heat reservoir and do useful work on the same object (e.g. piston or shaft), but work in counter phase. When one of these engines performs positive work in the process of isothermal expansion, the second one goes through the isothermal compression stage, and vice versa. Regeneration is done by controlled heat exchange. The first engine working body (WB1) with temperature T_H corresponding to the "hot" isotherm, at the beginning of isochoric cooling, is brought into thermal contact with the working body of the second engine (WB2) having at that moment the temperature of the "cold" isotherm T_C at the beginning of isochoric heating. Heat exchange takes place at a constant volume and during heat exchange WB1 cools down to some temperature T'_H and WB2 heats up to some temperature T'_C. At the same time heat exchange between WB1 (or WB2) and other bodies can be neglected. Then the thermal contact is broken, and WB1 cools down giving the heat to the environment while WB2 continues heating getting the heat from the heater. In the second half of the cycle the processes are repeated, only the WB1 and WB2 change roles. The WB1 and WB2 consist of the same amounts of gas, the gas may be considered ideal.

Let temperature T_H be higher than T_C by $\delta \equiv \dfrac{T_H - T_C}{T_H} = 28\,\%$ and the efficiency of the Stirling engine without regeneration be $\eta_0 = 21\,\%$. Let's call the ratio $r \equiv \dfrac{Q_R}{Q_+}$ the "regeneration factor", where Q_+ is the total heat obtained in one cycle by the WB of the engine that operates with regeneration.

1.1. At what values of T'_H and T'_C coefficient r will be the maximum possible for such regeneration scheme? Answer the question by expressing the sought temperatures in terms of T_H and T_C.

1.2. What is the maximum possible value of r? Write down the answer as a formula (that uses the quantities specified in the problem statement), and calculate it as a percentage accurate to a tenth.

1.3. Find the efficiency of the engine with maximum regeneration if the useful work equals $k = \dfrac{7}{8}$ of the total work performed by WB1 and WB2 (i.e. $1 - k = \dfrac{1}{8}$ gives the part of specified work lost on mechanical friction in the engine components, etc.). Write down the answer as a formula that uses the quantities specified in the problem statement, and calculate it in percentage rounding it to the nearest whole (if necessary).

Part II: Unusual Substance

Let some very unusual substance be at our disposal. We have the following information about this substance:

- its heat capacity in isobaric process C_p depends on absolute temperature $C_p = \beta(p)\cdot T$; the work performed by this substance and the amount of heat obtained by it in the isobaric process are related as $A = -\frac{1}{2}Q$;
- its heat capacity in isochoric process depends on temperature $C_V = \gamma(V)\cdot T^3$, and the equation of isochore is $p\cdot T^l = \text{const}$, where l is a constant exponential factor;
- if $T \to 0$ then volume and internal energy of this substance tend to zero at any finite pressure, and internal energy grows with temperature.

2.1. Write down the isobaric process equation for this substance in temperature-volume coordinates.

2.2. Write down (to a positive constant factor) the caloric equation of state of this substance, i.e. the equation connecting the internal energy with temperature and volume $U = U(V,T)$.

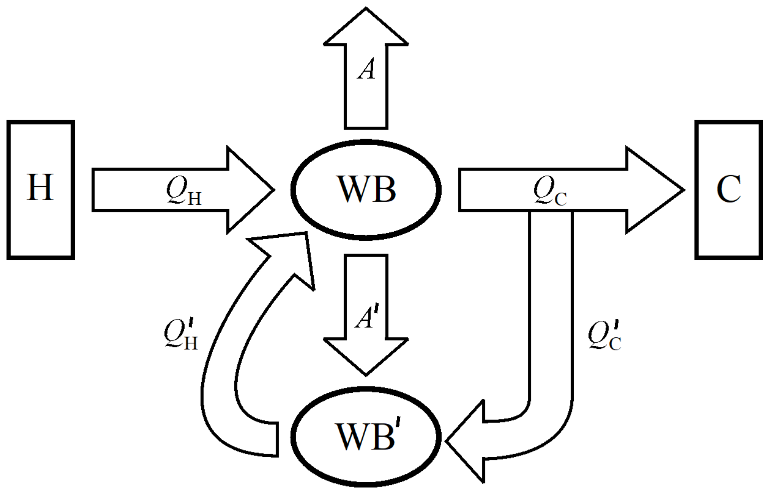
2.3. Write down (using the same positive constant factor) the thermal equation of state of this substance, i.e. the equation connecting pressure with temperature and volume $p = p(V,T)$.

Note: as for any thermodynamic system, the thermal and caloric equations of state for this substance satisfy $\left(\frac{\partial U}{\partial V}\right)_{T=const} = T\left(\frac{\partial p}{\partial T}\right)_{V=const} - p.$

2.4. Find the internal energy of this substance in terms of its pressure and volume. Write down the adiabatic equation of this substance in pressure-volume coordinates.

Part III: Regeneration using a Heat Pump

Let's consider another method of regeneration, when part of the work produced by the WB of heat engine is directed to the WB' of a heat pump. Due to this work, the heat pump returns a part of the heat given to the cooler by the WB during its cooling stage back to the WB during its heating stage. The scheme of operation of such a device is shown in the figure.



Let it contain a WB, which consists of constant amount of the same ideal gas as in Part I of the problem, performing the Stirling cycle with the same parameters (now we know that the ideal gas used as the WB is three-atomic). At the same time, the WB' is a constant amount of substance considered in Part II of the problem, which also performs the Stirling cycle. In this cycle, the ratio of the maximum absolute temperature to the minimum absolute temperature and the ratio of the maximum volume to the minimum volume are exactly the same as in the WB cycle. The efficiency of a heat pump is characterized by its *refrigerating factor* $\chi \equiv \frac{Q'_C}{A'}$, which is the ratio of the heat taken away from the colder body to the work spent on the activation of the heat pump.

3.1. Calculate the Stirling Cycle refrigerating factor with the substance from Part II and the parameters given in Part I. Give your answer in percentage rounded to the nearest whole.

3.2. Find the efficiency of an engine with this method of regeneration if the ratio of the amounts of substance is such that the regeneration factor $r \equiv \frac{Q'_H}{Q_+} = \frac{Q'_H}{Q_H + Q'_H} = 0{,}5$ and the useful work is $k = \frac{7}{8}$ part of work A. Give your answer in percentage rounded to the tenths.